

Area-scaling of quantum fluctuations

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We show that fluctuations of bulk operators that are restricted to some region of space scale as the surface area of the region, independently of its geometry. Specifically, we consider two point functions of operators that are integrals over local operator densities whose two point functions falls off rapidly at large distances, and does not diverge too strongly at short distances. We show that the two point function of such bulk operators is proportional to the area of the common boundary of the two spatial regions. Consequences of this, relevant to the holographic principle and to area-scaling of Unruh radiation are briefly discussed.

I. INTRODUCTION

The discovery of the non extensive nature of black hole entropy [1, 2] has lead to entropy bounds on matter, and to the proposal of the holographic principle (see [3] for a review) – a conjecture regarding the reduction of the number of degrees of freedom needed to describe a theory of quantum gravity. Later, it was discovered that other thermodynamic quantities of fields in a black-hole background also scale as the surface area of its horizon, and consequently, it was hypothesized that it is the entangled nature of the quantum state of the system inside and outside the horizon, which leads to area-scaling. This view has received some support from the area-scaling properties of entanglement entropy in flat space [4, 5, 6].

In this paper we study area-scaling of two point functions of a certain class of quantum field theory bulk operators in Minkowski space, to be defined shortly. We shall show, under weak assumptions, that quantum expectation values of operators restricted to a region of space will scale as its surface area, regardless of the region's geometry. This area-scaling property may then be used to establish area-scaling of thermodynamic quantities [7], and in some sense, a bulk-boundary correspondence [8].

The two point functions we wish to consider are the expectation values of a product of two operators which are restricted to some regions V_1 and V_2 of Minkowski space. In

order to restrict operators to a region V_i , we use operator densities $\mathcal{O}_j(\vec{x})$, and define $O_j^{V_i} = \int_{V_i} \mathcal{O}_j(\vec{x}) d^d x$.

We show that if the connected two point function of the operator densities $F(|\vec{x} - \vec{y}|)$, $\langle \mathcal{O}_i(\vec{x}) \mathcal{O}_j(\vec{y}) \rangle_C = F(|\vec{x} - \vec{y}|) \equiv \nabla^2 g$ satisfies the following conditions:

- (I) $g(\xi)$ is short range: at large distances it behaves like $g(\xi) \sim 1/\xi^a$, with $a \geq d - 1$,
- (II) $g(\xi)$ is not too singular at short distances. Explicitly, we require that for small ξ , $g(\xi) \sim 1/\xi^a$ with $a < d - 2$,

and if V_1 and V_2 are finite, then the connected two point function $\langle O_i^{V_1} O_j^{V_2} \rangle_C$ is proportional to the area of the common boundary of the two regions V_1 and V_2 : $\langle O_i^{V_1} O_j^{V_2} \rangle_C \propto S(B(V_1) \cap B(V_2))$. Here $B(V)$ is the boundary of V , and $S(B(V))$ is its area.

This implies that the fluctuations (or variance: $\text{var}(O_i^V) = \langle (O_i^{V^2} - \langle O_i^V \rangle^2)$) of the operator O_i^V scale as the surface area. In particular, the energy fluctuations inside V will be proportional to $S(B(V))$. This is discussed at length in [7].

In section II we give a detailed proof of the area-scaling property of two point functions of bulk operators satisfying conditions (I) and (II). Section III contains an explicit calculation of energy fluctuations, and fluctuations of the boost operator. We discuss these results in section IV.

II. AREA-SCALING OF TWO POINT FUNCTIONS

We shall first give a general explanation as to why conditions (I) and (II) are required for area-scaling, and outline the proof for area-scaling of two point functions. A more detailed discussion will then follow.

In order to evaluate $\langle O_i^{V_1} O_j^{V_2} \rangle_C$, we may express it as follows,

$$\begin{aligned} \langle O_i^{V_1} O_j^{V_2} \rangle_C &= \int_{V_1} \int_{V_2} F_{i,j}(|\vec{x} - \vec{y}|) d^d x d^d y \\ &= \int D(\xi) F_{i,j}(\xi) d\xi. \end{aligned} \tag{1}$$

The integral has been factored into a product of a purely geometric term $D(\xi)$, and an operator dependent term $F_{i,j}(\xi)$. Using $\nabla^2 g_{i,j}(\xi) \equiv F_{i,j}(\xi)$, we may integrate eq.(1) by parts. The surface term then vanishes due to conditions (I) and (II), and we get

$$\langle O_i^{V_1} O_j^{V_2} \rangle_C = - \int \frac{d}{d\xi} \left(D(\xi) \frac{1}{\xi^{d-1}} \right) \xi^{d-1} \frac{d}{d\xi} g(\xi) d\xi. \tag{2}$$

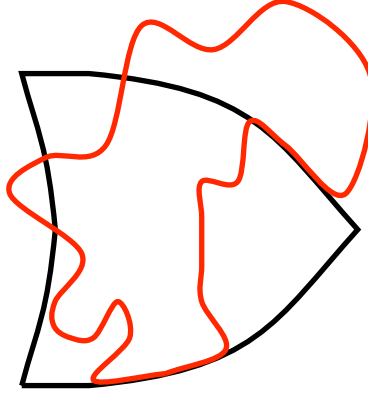


FIG. 1: A general case when the regions have some overlap with each other, and neither is fully contained in the other. The regions have common ‘boundaries from within’, and ‘boundaries from the outside’.

To proceed we note that the geometric term is of the form:

$$D_{V_1, V_2}(\xi) = \int_{V_1} \int_{V_2} \delta(\xi - |\vec{r}_1 - \vec{r}_2|) d^d r_1 d^d r_2. \quad (3)$$

We show in the next subsection that $D_{V_1, V_2} = G_V V \xi^{d-1} + G_S S \xi^d + \mathcal{O}(\xi^{d+1})$, with V a volume term, S a surface area term, and G numerical coefficients. There are some geometries for which this scaling is more obvious: for example, if V_1 and V_2 are disjoint volumes with a common boundary which may be approximated as flat, then one may carry out the integral in eq.(3) by switching to ‘center of mass coordinates’ $\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}$, and $\vec{r} = \vec{r}_2 - \vec{r}_1$. Restricting the value of $|\vec{r}|$ to be equal to ξ , results in confining \vec{R} to be, at most, a distance of $\xi/2$ from the boundary. The integral over the \vec{R} coordinate will give a term proportional to $S\xi$, S being the area of the boundary. The remaining integral over the \vec{r} coordinate will give a term proportional to ξ^{d-1} . Combining these results, we get the claimed form of D_{V_1, V_2} (the volume term vanishes in this case).

Another possibility is that $V_1 = V_2$. In this case an integral over the center of mass coordinate \vec{R} will yield, to leading order, a volume dependent term. The remaining integral over the \vec{r} coordinate will give the appropriate powers of ξ .

Using these two simple cases, one can now construct D_{V_1, V_2} for general geometries by dividing one of the volumes, say V_1 , into sub-volumes which are either contained in V_2 , just touching V_2 , or disconnected from V_2 . In order for the appropriate form of D_{V_1, V_2} to follow,

one would need that the coefficient G_S be exactly the same for both cases described above. This requires a more detailed calculation and is shown in the following subsection.

Going back to eq.(2), we observe that the integral comes from the region $\xi \rightarrow 0$, and that the contribution of the volume dependent term of D_{V_1, V_2} will vanish $\frac{\partial}{\partial \xi} \left(\frac{D(\xi)}{\xi^{d-1}} \right) \sim S + \mathcal{O}(\xi)$. The leading contribution to the integral will come from the surface term, which gives area-scaling behavior of two point functions, as claimed. The vanishing of the contribution of the volume term is a result of properties (I) and (II) of g , which make the surface term vanish when integrating eq.(1) by parts, and of the special polynomial dependence of $D_{V_1, V_2}(\xi)$ on ξ .

A. The geometric term.

As we have shown schematically, the area-scaling properties of correlations depend on the properties of the geometric term defined in (eq.(3)). In this subsection we shall study it in some detail.

First we note that there exists a ξ_{min} and ξ_{max} , such that $D(\xi) = 0$ for $\xi \geq \xi_{max}$ or $\xi \leq \xi_{min}$: define $\xi_{min} = \inf\{|\vec{x} - \vec{y}| | \vec{x} \in V_1, \vec{y} \in V_2\}$. For $\xi < \xi_{min}$ there are no values of \vec{x} and \vec{y} which will have a non zero contribution to the integral, and therefore $D(\xi) = 0$ for this region. Similarly $\xi_{max} = \sup\{|\vec{x} - \vec{y}| | \vec{x} \in V_1, \vec{y} \in V_2\}$.

For the cases where $\xi_{min} = 0$ (that is, the closed sets V_1 and V_2 are not disjoint), we shall show that $D_{V_1, V_2}(\xi)$ satisfies

$$D_{V_1, V_2}(\xi) = G_V V \xi^{d-1} + G_S (S(B_{in}) - S(B_{out})) \xi^d + \mathcal{O}(\xi^{d+1}).$$

G_V and G_S are constants which depend only on the dimensionality of space, explicitly $G_S = \frac{d \pi^{d/2}}{(d-1)\Gamma(d/2+1)}$. $B_{in/out}$ is the common boundary of V_1 and V_2 . B_{out} is the part of the boundary when V_1 and V_2 are on opposite sides of the boundary, whereas B_{in} is the part of the boundary when V_1 and V_2 are on the same side of the boundary. The first term is a volume dependent term.

To solve the integral (3) we switch to a ‘center of mass’ coordinate system (shown in Fig. 2): $\vec{R} = \frac{\vec{r}_1 + \vec{r}_2}{2}$, and $\vec{r} = \vec{r}_2 - \vec{r}_1$, so that \vec{r} points from \vec{r}_1 to \vec{r}_2 , and \vec{R} points to the middle of \vec{r} (since $\vec{R} = \vec{r}_1 + 1/2 \vec{r}$). In this new coordinate system,

$$D_{V_1, V_2}(\xi) = \int \int \delta(\xi - r) d^d R d^d r. \quad (4)$$

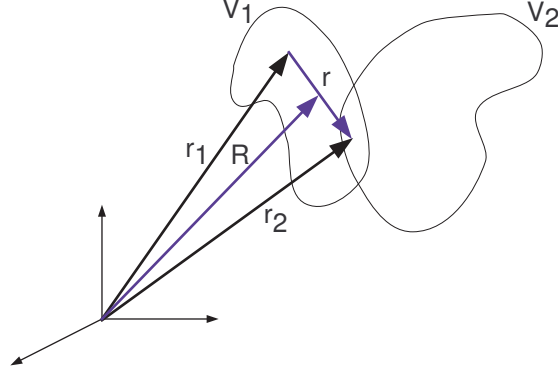


FIG. 2: A pictorial description of the (\vec{r}_1, \vec{r}_2) coordinates, and the (\vec{R}, \vec{r}) coordinates.

In order to carry out the integration, we wish to use a generalized ‘radial’ coordinate ρ such that $\rho = \rho_0$ will define the boundary $B(V_1)$ of V_1 , and generalized ‘angular’ coordinates α_i which will define solid angles on the boundary. To define such a coordinate system we foliate space into surfaces which, when very close to $B(V_1)$, are parallel to it. ρ is chosen to be the coordinate which points to different leaves of the foliation. $\rho = \rho_0$, as stated, defines the surface $B(V_1)$. We also choose ρ such that for a point \vec{R} for which $|\rho - \rho_0|$ is small enough, then $|\rho - \rho_0|$ will specify the distance of \vec{R} from the boundary. α_i are generalized angles on each hyper-surface. The unit volume in this coordinate system is

$$d^d R = J(\rho, \alpha_i) d\rho \prod_i d\alpha_i.$$

For the vector \vec{r} we choose a polar coordinate system:

$$d^d r = r^{d-1} d\Omega = r^{d-1} \sin^{d-2} \theta d\theta d\Omega_{\perp}.$$

For a given point \vec{R} , the integration over the angular coordinates of \vec{r} will give us the solid angle subtended by all allowed values of $\vec{r}_2 - \vec{r}_1$, when it is centered at \vec{R} . In preparation for performing the integral in eq.(4) for a general geometry (as shown in Fig. 1) we first consider several particular cases.

Case 1: Regions which have a common boundary, but no common interior. Here we expect to have

$$D_{V_1, V_2}(\xi) = G_S S(B(V_1) \cap B(V_2)) \xi^d + \mathcal{O}(\xi^{d+1}).$$

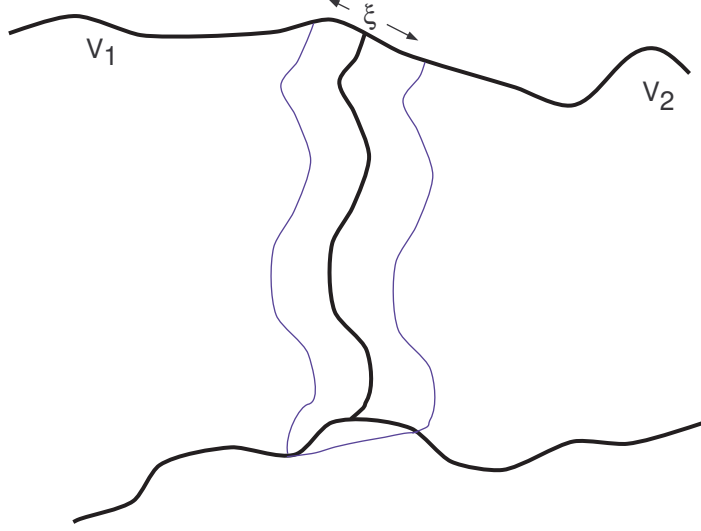


FIG. 3: Region of integration of the \vec{R} coordinate: The boundaries of regions 1 and 2 are given by the thick lines. The allowed region of \vec{R} is given by the thin lines.

Fixing ξ to have some small value, we look at all values \vec{r} is allowed to have for a fixed value of \vec{R} . Defining $\xi = |\vec{r}_1 - \vec{r}_2|$, then since \vec{r}_1 and \vec{r}_2 are on different sides of the boundary $B \equiv B(V_1) \cap B(V_2)$, \vec{R} is restricted to a distance of $\xi/2$ from B (See Fig.3). Therefore

$$\int d^d R = \int_{\rho_0 - \xi/2}^{\rho_0 + \xi/2} \int_{A(\rho)} J(\rho, \alpha_i) d\rho \prod_i d\alpha_i. \quad (5)$$

In this case $A(\rho)$ is the region of angular integration for each leaf of the foliation.

We denote the range of angles which define the common boundary as A_B , so that

$$B = \{(\rho_0, \alpha_i) | \alpha_i \in A_B\}.$$

We note that with this definition we may also write

$$\int_{A(\rho_0)} J(\rho_0, \alpha_i) \prod_i d\alpha_i = S(B) + \mathcal{O}(\xi).$$

For a certain point $\vec{R} = (\rho, \alpha_i)$ close to the boundary: $|\rho - \rho_0| < \xi/2$ (shown in Fig. 4), the angular integration over \vec{r} will be restricted to a region $\bar{\Omega}(\rho, \alpha_i; \xi)$.

Therefore we have:

$$D_{V_1, V_2}(\xi) = \int_{\rho_0 - \xi/2}^{\rho_0 + \xi/2} \int_{A(\rho)} J(\rho, \alpha_i) \int_{\bar{\Omega}(\rho, \alpha_i; \xi)} \xi^{d-1} d\Omega \prod_i d\alpha_i d\rho. \quad (6)$$

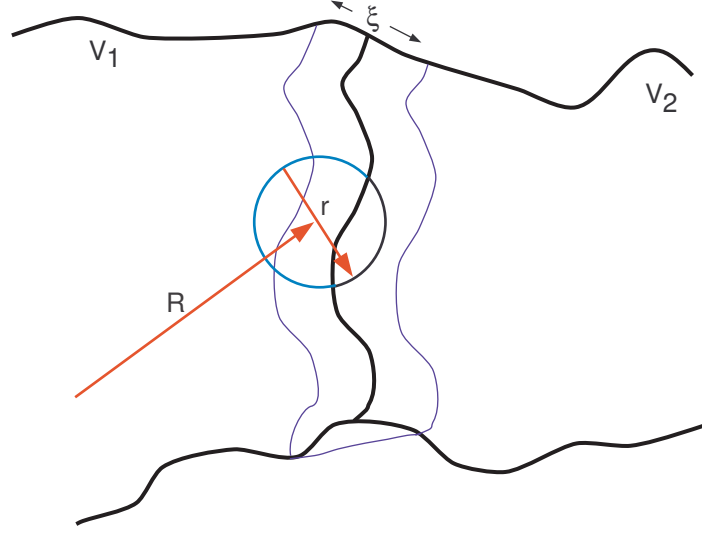


FIG. 4: The allowed region for \vec{r} in the case where the volumes are ‘just touching’. The dark end of the circle shows the region where \vec{r} is allowed to point to.

Changing variables to the dimensionless variable $\zeta = (\rho - \rho_0)/(\xi/2)$, and integrating over the angular coordinate Ω we obtain,

$$D_{V_1, V_2}(\xi) = \xi^{d-1} \int_{-1}^1 \int_{A(\rho_0 + \xi/2\zeta)} J(\rho_0 + \xi/2\zeta, \alpha_i) \bar{\Omega}(\rho_0 + \xi/2\zeta, \alpha_i; \xi) \prod_i d\alpha_i \frac{\xi}{2} d\zeta. \quad (7)$$

This is an exact expression.

Expanding the integrand on the r.h.s. of eq.(7) in powers of ξ , we obtain

$$\begin{aligned} & \int_{A(\rho_0 + \xi/2\zeta)} J(\rho_0 + \xi/2\zeta, \alpha_i) \bar{\Omega}(\rho_0 + \xi/2\zeta, \alpha_i; \xi) \prod_i d\alpha_i \\ &= \int_{A(\rho_0 + \xi/2\zeta)} J(\rho_0 + \xi/2\zeta, \alpha_i) \bar{\Omega}(\rho_0 + \xi/2\zeta, \alpha_i; \xi) \prod_i d\alpha_i \Big|_{\xi=0} + \mathcal{O}(\xi) \\ &= \int_{A(\rho_0)} J(\rho_0, \alpha_i) \Omega(\zeta) \prod_i d\alpha_i + \mathcal{O}(\xi), \end{aligned} \quad (8)$$

where

$$\Omega(\zeta) = \lim_{\xi \rightarrow 0} \bar{\Omega}(\rho_0 + \xi/2\zeta, \alpha_i; \xi).$$

We will show that Ω is a function of ζ only, and obtain an explicit expression for it.

For very small ξ , we look at a point \vec{R} , a distance $|\rho - \rho_0| < \xi/2$ from the boundary, and calculate the solid angle subtended by a vector \vec{r} centered at \vec{R} whose one end is in V_1 , and

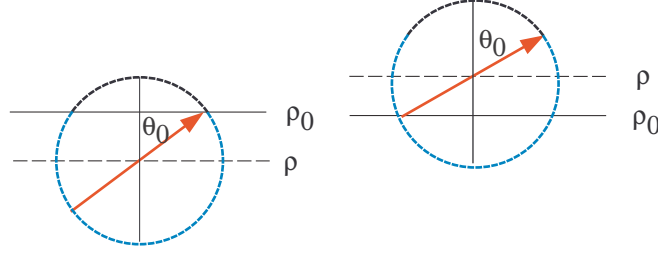


FIG. 5: The allowed region for \vec{r} (dark region) in the case where ξ is very small, so that the boundary may be approximated as flat. ρ_0 specifies the location of the common boundary, and ρ the leaf of the foliation to which \vec{R} is pointing to. The left diagram is for the case where $\vec{R} \in V_1$, and the right diagram for $\vec{R} \in V_2$.

other end is in V_2 . Since we are taking the limit where $\xi \rightarrow 0$, the shape of the boundary close to the point \vec{R} may be considered flat (shown in Fig. 5). Therefore Ω is the solid angle generated by \vec{r} when very close to a flat surface.

Defining the z-axis as the axis perpendicular to the surface, we get that the angle θ between the z-axis and \vec{r} can range over values from 0 to θ_0 , with $\cos(\theta_0) = (\rho_0 - \rho)/(\xi/2)$ for $\rho \leq \rho_0$ and $\cos(\theta_0) = (\rho - \rho_0)/(\xi/2)$ for $\rho \geq \rho_0$. Note that in any case $0 \leq \theta_0 \leq \pi/2$, implying that $\cos \theta_0 \geq 0$.

Therefore, for $d > 2$,

$$\Omega = \int_0^{\theta_0} \sin^{d-2} \theta d\theta \int d\Omega_{\perp}.$$

We note that

$$\int_0^{\theta_0} \sin^{d-2} \theta d\theta = \frac{1}{2} \left(B\left(\frac{1}{2}, \frac{d-1}{2}\right) - B_{\zeta^2}\left(\frac{1}{2}, \frac{d-1}{2}\right) \right) \quad (9)$$

where $B_x(a, b)$ is the partial beta function. The full solid angle of a d-dimensional sphere is given by

$$\begin{aligned} C_d &= \int_0^{\pi} \sin^{d-2} \theta d\theta \int d\Omega_{\perp} \\ &= B\left(\frac{1}{2}, \frac{d-1}{2}\right) \int d\Omega_{\perp}. \end{aligned} \quad (10)$$

Putting eq.(10) into eq.(8), we get

$$\int_{A(\rho_0)} J(\rho_0, \alpha_i) \frac{C_d}{2} \left(1 - \frac{B_{\zeta^2}(\frac{1}{2}, \frac{d-1}{2})}{B(\frac{1}{2}, \frac{d-1}{2})} \right) \prod_i d\alpha_i + \mathcal{O}(\xi).$$

Integrating over the angular coordinates we are left with

$$\frac{C_d}{2} \left(1 - \frac{B_{\zeta^2(\frac{1}{2}, \frac{d-1}{2})}}{B(\frac{1}{2}, \frac{d-1}{2})} \right) S(B) + \mathcal{O}(\xi),$$

and plugging this into eq.(7), will give us:

$$\begin{aligned} D_{V_1, V_2}(\xi) &= \frac{\xi}{2} \xi^{d-1} \left[\frac{C_d}{2} \int_{-1}^1 \left(1 - \frac{B_{\zeta^2(\frac{1}{2}, \frac{d-1}{2})}}{B(\frac{1}{2}, \frac{d-1}{2})} \right) S(B) d\zeta + \mathcal{O}(\xi) \right] \\ &= \frac{\xi^d}{2} C_d S(B) \int_0^1 \left(1 - \frac{B_{\zeta^2(\frac{1}{2}, \frac{d-1}{2})}}{B(\frac{1}{2}, \frac{d-1}{2})} \right) d\zeta + \mathcal{O}(\xi^{d+1}). \end{aligned} \quad (11)$$

So now we have to evaluate the integral

$$\int_0^1 B_{\zeta^2} \left(\frac{1}{2}, \frac{d-1}{2} \right) d\zeta = \int_0^1 \int_0^{\zeta^2} t^{-\frac{1}{2}} (1-t)^{\frac{d-3}{2}} dt d\zeta,$$

which can be done by changing the order of integration. The result is that

$$D_{V_1, V_2}(\xi) = \xi^d \frac{C_d}{2} S(B) \frac{B(1, \frac{d-1}{2})}{B(\frac{1}{2}, \frac{d-1}{2})} + \mathcal{O}(\xi^{d+1}).$$

Expressing the Beta function as a product of Gamma functions, and using the explicit expression $C_d = \frac{d\pi^{d/2}}{\Gamma(d/2+1)}$, we get the final result

$$D_{V_1, V_2}(\xi) = \frac{d\pi^{d/2}}{(d-1)\Gamma(d/2+1)} S(B) \xi^d + \mathcal{O}(\xi). \quad (12)$$

Case 2: Next we consider geometries where $V_1 = V_2 \equiv V$, for which we expect to have:

$$D_{V_1, V_2}(\xi) = G_V V \xi^{d-1} - G_S S(B(V)) \xi^d + \mathcal{O}(\xi^{d+1}).$$

We consider the same foliation of space as before. In this case, even at small ξ , we integrate over points inside V . The angular integration over \vec{r} will be constrained only when the region where \vec{R} is a distance of $\xi/2$ from the boundary. Therefore:

$$\begin{aligned} D_{V_1, V_2}(\xi) &= \int_0^{\rho_0} \int_{A(\rho)} \int_{\bar{\Omega}} \xi^{d-1} J(\rho, \alpha_i) d\Omega \prod_i d\alpha_i d\rho \\ &= \int_0^{\rho_0 - \xi/2} \int_{A(\rho)} \xi^{d-1} J(\rho, \alpha_i) C_d \prod_i d\alpha_i d\rho \\ &\quad + \int_{\rho_0 - \xi/2}^{\rho_0} \int_{A(\rho)} \int_{\bar{\Omega}} \xi^{d-1} J(\rho, \alpha_i) d\Omega \prod_i d\alpha_i d\rho. \end{aligned} \quad (13)$$

The first expression is the unconstrained part, and the second expression is the constrained part.

Considering first the integral for which \vec{r} is not constrained, we have

$$\begin{aligned} \int_0^{\rho_0 - \xi/2} \int_{A(\rho)} \xi^{d-1} J(\rho, \alpha_i) C_d \prod_i d\alpha_i d\rho \\ = VC_d \xi^{d-1} - \int_{\rho_0 - \xi/2}^{\rho_0} \int_{A(\rho)} \xi^{d-1} J(\rho, \alpha_i) \int_0^\pi \sin^{d-2} \theta d\theta d\Omega_\perp \prod_i d\alpha_i d\rho. \end{aligned}$$

Proceeding as before, we change variables of integration to $\zeta = (\rho - \rho_0)/(\xi/2)$, and expand in small ξ .

$$VC_d \xi^{d-1} - S(B) \frac{\xi^d}{2} \int_{-1}^0 \int_0^\pi \sin^{d-2} \theta d\theta d\Omega_\perp d\zeta + \mathcal{O}(\xi^{d+1}). \quad (14)$$

Under the same approximation, the constrained part reduces to

$$S(B) \frac{\xi^d}{2} \int_{-1}^0 \Omega(\zeta) d\zeta + \mathcal{O}(\xi^{d+1}). \quad (15)$$

In this case, θ , the angle between \vec{r} and the z-axis, is restricted to θ_0 , and $\pi - \theta_0$ where $\cos \theta_0 = -\zeta$. Hence

$$\Omega(\zeta) = \int_{\theta_0}^{\pi - \theta_0} \sin^{d-2} \theta d\theta.$$

Combining the contributions of the surface terms of (14) and (15) to those of the surface term of (13), we have

$$-S(B) \frac{\xi^d}{2} \int_{-1}^0 \left(\int_0^\theta + \int_{\pi - \theta}^\pi \right) \sin^{d-2} \theta d\theta \int \Omega_\perp d\zeta.$$

Since $0 \leq \theta \leq \pi/2$, both angular integrals are equal, and the above equation simplifies to

$$-S(B) \frac{\xi^d}{2} \int_{-1}^0 2 \int_0^\theta \sin^{d-2} \theta d\theta \int \Omega_\perp d\zeta.$$

Using eq.(9) we may carry out the integral over the θ coordinate:

$$\begin{aligned} -S(B) \frac{\xi^d}{2} 2 \int_{-1}^0 \frac{1}{2} \left(B\left(\frac{1}{2}, \frac{d-1}{2}\right) - B_{\zeta^2}\left(\frac{1}{2}, \frac{d-1}{2}\right) \right) d\zeta \\ = -S(B) \frac{\xi^d}{2} \int_{-1}^1 \frac{1}{2} \left(B\left(\frac{1}{2}, \frac{d-1}{2}\right) - B_{\zeta^2}\left(\frac{1}{2}, \frac{d-1}{2}\right) \right) d\zeta. \end{aligned}$$

Comparing this with eq.(11), we find that the surface term contribution has exactly the same magnitude as the leading order contribution to $D_{V_1, V_2}(\xi)$ in Case 1, its sign, however,

is negative.

Case 3: V_1 is fully contained in V_2 with no common boundaries. Here we shall have:

$$D_{V_1, V_2} = C_d V_1 \xi^{d-1} + \mathcal{O}(\xi^{d+1}).$$

Consider V_1 and its complement in V_2 : $V_2 \setminus V_1$. We may write $D_{V_1, V_2}(\xi) = D_{V_1, V_1}(\xi) + D_{V_1, V_2 \setminus V_1}(\xi)$. V_1 and $V_2 \setminus V_1$ are disjoint sets with a common boundary, therefore they satisfy the conditions of Case 1. Applying the results of Case 1 to V_1 and $V_2 \setminus V_1$ we get

$$\begin{aligned} D_{V_1, V_2 \setminus V_1}(\xi) &= G_S \xi^d S(B(V_1) \cap B(V_2 \setminus V_1)) + \mathcal{O}(\xi^{d+1}) \\ &= G_S \xi^d S(B(V_1)) + \mathcal{O}(\xi^{d+1}). \end{aligned}$$

Using Case 2 to calculate $D_{V_1, V_1}(\xi)$, we find that the surface terms in $D_{V_1, V_1}(\xi) + D_{V_1, V_2 \setminus V_1}(\xi)$ exactly cancel each other.

Case 4: V_1 is contained in V_2 and they do have a common boundary. In this case the result is that

$$D_{V_1, V_2} = C_d V_1 \xi^{d-1} - G_S S(B(V_1) \cap B(V_2)) \xi^d + \mathcal{O}(\xi^{d+1}).$$

Defining the common boundary of V_1 and V_2 as $B \equiv B(V_1) \cap B(V_2)$, we consider V_3 , the complement of V_2 . Since V_2 and V_3 have the same boundary then $B(V_1) \cap B(V_2) = B(V_1) \cap B(V_3) = B$. Since the interior of V_3 is disjoint from the interior of V_1 ($V_1 \subset V_2$), it satisfies the conditions for Case 1, so that $D_{V_1, V_3}(\xi) = G_S S(B(V_1) \cap B(V_2)) \xi^d + \mathcal{O}(\xi^{d+1})$. Using Case 3, we also have that $D_{V_1, V_2 \cup V_3}(\xi) = C_d V_1 + \mathcal{O}(\xi^{d+1})$, so that $D_{V_1, V_2}(\xi) = C_d V_1 - D_{V_1, V_3}(\xi) + \mathcal{O}(\xi^{d+1})$, which gives the deired result.

Case 5: V_1 and V_2 have no common boundary and are not disjoint.

$$D_{V_1, V_2} = C_d (V_2 \cap V_1) \xi^{d-1} + \mathcal{O}(\xi^{d+1}).$$

We have already seen that this is correct if V_1 is fully contained in V_2 . What is left, is to check the case when only part of V_1 is contained in V_2 . In this case we may define $V_{2,in} = V_2 \cap V_1$, and $V_{2,out} = V_2 \setminus V_{2,in}$ (so that $V_2 = V_{2,in} \cup V_{2,out}$). The boundary of $V_{2,in}$

has a common boundary with V_1 . Since $V_{2,out}$ is the complement of $V_{2,in}$ relative to V_2 , any boundary of $V_{2,out}$ which is not a boundary of V_2 is also a boundary of $V_{2,in}$, and since V_2 does not have a common boundary with V_1 , any boundary of $V_{2,in}$ which is common to V_1 must also be common to $V_{2,out}$. Therefore to leading order,

$$\begin{aligned} D_{V_1,V_2} &= D_{V_1,V_{2,in}} + D_{V_1,V_{2,out}} \\ &= C_d V_{2,in} \xi^{d-1} + \frac{C_{d-1}}{d-1} (S(B(V_{2,out}) \cap B(V_1)) - S(B(V_{2,in}) \cap B(V_1))) \xi^d \\ &= C_d (V_2 \cap V_1) \xi^{d-1}. \end{aligned}$$

We are using here somewhat imprecise notation as we are not differentiating between the set $V_1 \cap V_2$ and its volume.

General geometries: In order to avoid problems with infinities we consider:

1. Regions V_1 and V_2 whose common boundary has a finite number of connected components.
2. Both volumes are connected [19].

An example of regions satisfying these conditions is given in Fig. 1.

The idea is to divide V_2 into subsets, such that each subset of V_2 will either contain a single connected subset of the boundary of V_1 which is common to V_2 , or no such boundary at all: consider a single connected part of $B(V_1) \cap B(V_2)$, which we denote by B . We shall construct a volume $V'_2 \subseteq V_2$ such that (a) $B = B(V_1) \cap B(V'_2)$ and (b) V'_2 has no other common boundaries with V_1 , and does not contain a boundary of V_1 .

This construction is achieved by first enclosing B within a volume A such that A does not contain, and is far enough from, any other boundary of V_1 . Then we define $V'_2 = A \cap V_2$. That $V'_2 \subseteq V_2$ is obvious from the definition. Condition (a) is satisfied since B is common to both $B(V_2)$ and $B(A)$. Condition (b) is satisfied since if V'_2 has another boundary with V_1 or contains a boundary of V_1 then this must be contained in A or a boundary of A , which is a contradiction.

This procedure can be applied consecutively to all common components of the boundary of V_1 and V_2 , in such a way that all the V'_2 are disjoint (if they are not disjoint we may get rid of their common part by a redefinition). Thus we obtain a finite collection of sub-volumes

V_{2_i} that are contained in V_2 , and that satisfy $(\bigcup_i B(V_{2_i})) \cap B(V_1) = B(V_2) \cap B(V_1)$. In words: the common boundary of the V_{2_i} 's with V_1 is equal to the common boundary of V_2 and V_1 .

We also define $V_{2,bulk} = V_2 \setminus \bigcup_i V_{2_i}$, so that $V_2 = \bigcup_i V_{2_i} \cup V_{2,bulk}$. $V_{2,bulk}$ will have no common boundary with V_1 . This follows from observing that if it does have a common boundary with V_1 then this boundary must not be a boundary of V_2 with V_1 since all of these appear in the V_{2_i} 's. Since $V_{2,bulk}$ is the complement of $\bigcup_i V_{2_i}$ with respect to V_2 , then $\bigcup_i V_{2_i}$ must also have a boundary common to V_1 which is not common to V_2 , but this is not allowed by construction.

Now, $D_{V_1, V_{2,bulk}} \propto V_1 \cap V_{2,bulk} + \mathcal{O}(\xi^{d+1})$ since it satisfies the conditions in Case 5. Also we note that for each i , the interior of V_{2_i} is either contained in V_1 or disjoint from V_1 —an observation which follows from the fact that no interior of V_{2_i} contains a boundary of V_1 and so cannot cross from the interior to the exterior of V_1 .

This implies that for indices i for which $V_{2_i} \subseteq V_1$, we get

$$\begin{aligned} \sum_{i \in in} S(B(V_{2_i}) \cap B(V_1)) &= S\left(\bigcup_{i \in in} (V_{2_i} \cap B(V_1))\right) \\ &= S\left(\left(\bigcup_{i \in in} V_{2_i}\right) \cap B(V_1)\right) \\ &\equiv S_{in} \end{aligned}$$

Where S_{in} is the surface area of the common boundary of V_2 and V_1 for which, close to the boundary, the interiors are not disjoint. S_{out} is similarly defined.

We also note, that using case 4, $D_{V_1, V_{2_i}} \sim V_{2_i} \pm S(B(V_{2_i}) \cap B(V_1))$, for $i \in in$ (or $i \in out$). Therefore, the leading order behavior of $D_{V_1, V_2} = \sum_i D_{V_1, V_{2_i}} + D_{V_1, V_{2,bulk}}$ is

$$C_d V_{2,bulk} \cap V_1 \xi^{d-1} + \frac{C_{d-1}}{d-1} \left(\sum_{i \in in} (V_{2_i} \cap V_1 + S(B(V_{2_i}) \cap B(V_1))) - \sum_{j \in out} (S(B(V_{2_j}) \cap B(V_1))) \right) \xi^d,$$

which reduces to

$$C_d V_2 \cap V_1 \xi^{d-1} + \frac{C_{d-1}}{d-1} (S_{in} - S_{out}) \xi^d.$$

This completes the proof of our claim regarding the leading order behavior of the geometric term $D_{V_1, V_2}(\xi)$.

B. Area-scaling of two point functions.

Going back to eq.(1), we can now evaluate:

$$\langle O_i^{V_1} O_j^{V_2} \rangle_C = \int_{\xi_{min}}^{\xi_{max}} D(\xi) F(\xi) = \int_{\xi_{min}}^{\xi_{max}} D(\xi) \nabla^2 g d\xi.$$

First we integrate by parts (See eq.(2)),

$$\begin{aligned} \langle O_i^{V_1} O_j^{V_2} \rangle_C &= \int D(\xi) \frac{1}{\xi^{d-1}} \frac{d}{d\xi} \xi^{d-1} \frac{d}{d\xi} g(\xi) d\xi \\ &= D(\xi) \frac{d}{d\xi} g(\xi) \Big|_{\xi_{min}}^{\xi_{max}} - \int \frac{d}{d\xi} \left(D(\xi) \frac{1}{\xi^{d-1}} \right) \xi^{d-1} \frac{d}{d\xi} g(\xi) d\xi. \end{aligned}$$

Consider the surface term. For finite and non zero ξ_{min} and ξ_{max} , D vanishes and therefore the surface term vanishes. When $\xi_{min} = 0$, we note that at small ξ , since $D(\xi) \sim V \xi^{d-1} + \mathcal{O}(\xi^d)$, then

$$Dg' \sim V \xi^{d-1} \frac{1}{\xi^{a-1}} + \mathcal{O}(\xi^{d-a+1}),$$

and because of condition (I) this term vanishes. We shall see that the limit $\xi_{max} \rightarrow \infty$ poses a special problem, and may require the introduction of a long-distance (IR) cutoff.

We now have:

$$\langle O_i^{V_1} O_j^{V_2} \rangle_C = - \int_{\xi_{min}}^{\xi_{max}} \tilde{D}(\xi) \xi^{d-1} g'(\xi) d\xi,$$

where we have defined $\tilde{D}(\xi) = \frac{d}{d\xi} \left(D(\xi) \frac{1}{\xi^{d-1}} \right)$. Since \tilde{D} is constant for small ξ then in order for $\langle O_i^{V_1} O_j^{V_2} \rangle_C$ to converge at the lower limit, we need that condition (II) be satisfied.

Introducing a short-distance (UV) cutoff scale Λ ,

$$\begin{aligned} \langle O_i^{V_1} O_j^{V_2} \rangle_C &= - \int_{\xi_{min}}^{\xi_{max}} \Lambda^{-(d-1)} \hat{D}(\Lambda\xi) \Lambda^{-(d-1)} (\Lambda\xi)^{d-1} \Lambda^\alpha g'(\Lambda\xi) \Lambda^{-1} d\Lambda\xi \\ &= - \int_{\xi_{min}}^{y_{max}} \Lambda^{-2d+1+\alpha} \hat{D}(y) y^{d-1} g'(y) dy, \end{aligned} \tag{16}$$

where α is the scaling dimension of g' : $\Lambda^\alpha g'(\Lambda\xi; \Lambda^t m, 1) = g'(\xi; m, \Lambda)$. Here we have introduced explicitly the parameter ‘ m ’ to allow for the possibility that the theory contains other dimensionful parameters in addition to Λ (such as masses, or an IR cutoff). For simplicity, we have introduced a single such parameter. \hat{D} is dependent on the (dimensionful) geometric parameters of V_1 and V_2 . An explicit expression for α is obtained by noting that if $\mathcal{O}_i(\vec{x})$ has scaling dimension δ_i , then F has dimension $\delta_i + \delta_j + 2d$, so that g' has dimension

$\alpha = \delta_i + \delta_j + 2d - 1 \equiv \delta + 2d - 1$. Also, \tilde{D} scales as $[\text{length}]^{d-1}$, so that the geometric parameters in \hat{D} are now scaled to a length $1/\Lambda$.

Taking the shortest scale in the problem to be $1/\Lambda$, we wish to take the limit $\Lambda\xi \rightarrow \infty$. When both regions are disjoint and $\xi_{min} \neq 0$, we find that the integral in eq.(16) vanishes. This is supported by numerical results where it is seen that disjoint volumes have zero covariance as long as the distance between them is larger than the UV scale [9]. If the regions have some overlap or a common boundary then $\xi_{min} = 0$. Since $\tilde{D}(\xi)$ is constant at small ξ then $\hat{D}(y)$ will be constant for all but very large y . This will then allow us to evaluate $\langle O_i^{V_1} O_j^{V_2} \rangle_C$ as

$$\langle O_i^{V_1} O_j^{V_2} \rangle_C = \Lambda^\delta \int_0^\infty \hat{D}(y) y^{d-1} g'(y) dy + \mathcal{O}(1/y_{max})$$

as long as there are no contributions from the ‘rescaled infinity’. Writing out $\tilde{D}(\xi) = \sum_{n=0} d_n \xi^n$, or $\hat{D}(y) = \sum_{n=0} d_n / \Lambda^{-(d-1)+n} y^n$, we get that

$$\langle O_i^{V_1} O_j^{V_2} \rangle_C = -\Lambda^\delta \Lambda^{d-1} d_0 \int_0^\infty y^{d-1} g'(y) dy + \mathcal{O}(1/\Lambda \xi_{max}, d_n / \Lambda^{-(d-1)+n}, \Lambda^t m). \quad (17)$$

d_0 is the mutual surface area. We have established that $\langle O_i^{V_1} O_j^{V_2} \rangle_C$ scales linearly with the area of the common boundary, to leading order in the geometric parameters. The remaining terms in eq.(17) that contain additional geometric parameters, such as ξ_{max} , and d_n (for $n > 0$), are subleading and scale with a smaller power of the area. For example, if the spatial regions are d -dimensional spheres of radii R then the subleading terms scale as R^a with $a < d - 1$.

We would like to point out that the leading power of Λ in the expansion of $\langle O_i^V O_j^V \rangle_C$ may appear in one of the subleading terms, where it would appear multiplied by one of the other dimensionful parameters which we have denoted by m . Indeed, introducing an IR scale L in the theory may spoil the area-scaling behavior. We shall show an example of this in the next section.

III. EXPLICIT CALCULATIONS.

We have carried out several independent calculations of the energy fluctuations, and ‘boost generator fluctuations’ of a free massive scalar field in various volumes [7, 9, 10, 11, 12]. We present some of these calculations here.

A. The two point function of the energy operator

As an explicit example, we wish to find the two point function for the energy operator in a volume V of Minkowski space for a free massless scalar field. This is given by

$$\begin{aligned} \langle : E^{V_1} :: E^{V_2} : \rangle = & \frac{1}{8} \frac{1}{(2\pi)^{2d}} \int \int \int_{V_1} \int_{V_2} \left(\frac{-\vec{p} \cdot \vec{q}}{\sqrt{\omega_p \omega_q}} - \sqrt{\omega_p \omega_q} \right)^2 \\ & \times e^{i(\vec{p} + \vec{q}) \cdot (\vec{x} - \vec{y})} d^d p d^d q d^d x d^d y, \end{aligned} \quad (18)$$

which may be written in the form $\int_0^\infty F(\xi) D(\xi) d\xi$, where for a free field theory,

$$F(x) = \frac{1}{8} \frac{1}{(2\pi)^{2d}} \int \left(pq + 2\vec{p} \cdot \vec{q} + \frac{(\vec{p} \cdot \vec{q})^2}{pq} \right) e^{-i(\vec{p} + \vec{q}) \cdot \vec{x}} d^d p d^d q. \quad (19)$$

To find an expression for $F(\xi)$, we switch to a coordinate system where:

$$\vec{x} = \begin{pmatrix} x \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \quad \vec{q} = \begin{pmatrix} q_x \\ q_\perp \\ 0 \\ \vdots \\ 0 \end{pmatrix}; \quad \vec{p} = \begin{pmatrix} p_x \\ p_\perp \cos \theta_p \\ p_{\perp\perp} \\ \vdots \\ 0 \end{pmatrix}.$$

In this form, we may do all angular integrations:

$$\begin{aligned} F(x) = & \frac{1}{8} \frac{1}{(2\pi)^{2d}} \left(\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2} + 1)} (d-1) \right)^2 \\ & \times \int \left(pq + 2p_x q_x + \frac{p_x^2 q_x^2}{pq} + \frac{p_\perp^2 q_\perp^2}{pq} \frac{1}{d-1} \right) \\ & \times e^{-i(p_x + q_x)x} p_\perp^{d-2} q_\perp^{d-2} dp_\perp dq_\perp dp_x dq_x. \end{aligned}$$

Switching to polar coordinates in the remaining two dimensional system:

$$p_\perp = p \sin \theta$$

$$p_x = p \cos \theta,$$

and noting that integrations over the p and q variables are independent, we can now evaluate the integral by imposing an exponential cutoff $C(p/\Lambda) = e^{-p/\Lambda}$.

$$\begin{aligned} F(x) = & \frac{(d+1)\Gamma\left(\frac{d+1}{2}\right)\Lambda^{2(d+1)}}{8\pi^{d+1}(1+(\Lambda x)^2)^{d+3}} (d - 2(d+2)(\Lambda x) + d(\Lambda x)^4) \\ = & \frac{(d+1)\Gamma\left(\frac{d+1}{2}\right)\Lambda^{2(d+1)}}{8\pi^{d+1}} \nabla_{\Lambda x}^2 \frac{(\Lambda x)^2 - 1}{2(d+2)(1+(\Lambda x)^2)^{d+1}}. \end{aligned} \quad (20)$$

Therefore, for volumes with a common boundary B,

$$\langle : E^{V_1} :: E^{V_2} : \rangle \approx -\frac{(d+1)\Gamma^2\left(\frac{d+1}{2}\right)\Lambda^{d+1}}{8\pi^{d+1}}\frac{C_{d-1}}{d-1}\int_0^\infty y^{d-1}\frac{\partial}{\partial y}\frac{y^2-1}{2(d+2)(1+y^2)^{d+1}}dy.$$

Doing this integral we get:

$$\langle : E^{V_1} :: E^{V_2} : \rangle \approx -\frac{\Gamma\left(\frac{d+1}{2}\right)\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(2+\frac{d}{2}\right)}\frac{\Lambda^{d+2}}{2^{d+4}\pi^{\frac{d}{2}+1}}(S(B_{out})-S(B_{in})).$$

B. The Boost operator for half of Minkowski space.

As another exercise, we calculate the variance of two boost operators, when the volume in question is half of Minkowski space. We start with fluctuations of the boost generator in the 'z' direction, $B^{(z)V}$, where [13]

$$B^{(z)V} = \int_V z \mathcal{H} d^d x.$$

Now,

$$\langle : (B^{(z)V})^2 : \rangle = \int z_1 z_2 F_{E,E}(|\vec{x}_1 - \vec{x}_2|) d^d x_1 d^d x_2.$$

In this case it will be more useful to calculate $D(\xi)$ explicitly:

$$D_d(\xi) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \int_0^{\infty} \int_0^{\infty} \delta^{(d)}(\xi - |\vec{x}_1 - \vec{x}_2|) d^d x_1 d^d x_2.$$

Switching to $\vec{r}_{\pm} = \vec{x}_1 \pm \vec{x}_2$ coordinates, we may integrate over the transverse \vec{r}_{\pm} directions, yielding the transverse volume V_{\perp} (Transverse meaning the direction transverse to the z coordinate.) Therefore:

$$D_d(\xi) = \frac{1}{2} V_{\perp} \int_0^{\infty} \int_{-z_+}^{z_+} \left[\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \delta^{(d)}(\xi - r_-) d^{d-1} r_{-\perp} \right] dz_- dz_+.$$

Hence

$$\langle : (B^{(z)V})^2 : \rangle = V_{\perp} \int_0^{\infty} \int_{-z_+}^{z_+} \left[\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \frac{1}{4} (z_+^2 - z_-^2) F_d(r_-) d^{d-1} r_{-\perp} \right] dz_- dz_+, \quad (21)$$

where $F(r_-)$ is defined in eq.(19) and is evaluated in eq.(20).

By integrating by parts, this integral may be carried out exactly

$$\langle : (B^{(z)V})^2 : \rangle = \frac{1}{\Lambda^2} \frac{1}{d-1} \langle : E_V :^2 \rangle.$$

Here, again, the fluctuations are proportional to the surface area.

When we consider boosts in the directions parallel to the boundary, we need to deal with the IR scale ‘ L ’ (which needs to be introduced in order to define the boundary). Treating the IR scale as a dimensional parameter of dimension -1, we see that according to (17) it may contribute a factor of L to the fluctuations. It is a simple matter to generalize equation (21) to boosts in the other directions, yielding

$$\langle : (B^{(\perp)V})^2 : \rangle = V_{\perp} L^2 \Lambda^{d+1} \frac{(d+1) \Gamma^2\left(\frac{d+1}{2}\right)}{16^{d+6} \pi^{\frac{d}{2}+1} \Gamma\left(2 + \frac{d}{2}\right)} + \mathcal{O}(V_{\perp}).$$

We get that the fluctuations are not proportional to the surface area. As stated earlier this is due to the IR cutoff we imposed on directions parallel to the boundary of half of space.

IV. DISCUSSION

We have shown that under conditions (I) and (II), two point functions of bulk operators restricted to some regions of Minkowski space will scale as the surface area of the common boundary of the two regions, independently of their geometries.

Generally, one would expect that fluctuations, quantum or statistical, scale as the volume in which they are being measured, and not as the surface area of the volume. In a thermodynamic context, fluctuations of observables represent thermodynamic quantities. Energy fluctuations for example, correspond to heat capacity, which is usually extensive.

We have found instead, that fluctuations scale as the area of the region of space in which they are being measured. We can give a thermodynamic interpretation to the quantum fluctuations that we have calculated by considering not an observer making quantum measurements inside the volume V , but a different observer who has no access to the region outside V . If the initial state of the system in the whole space is $|\psi\rangle$, then an observer that has no access to the region outside V will see a state described by the density matrix $\rho_{in} = \text{Trace}_{out} |\psi\rangle\langle\psi|$. It is possible to show [14] that for any operator O^V which acts only inside the region V , $\langle\psi|O^V|\psi\rangle = \text{Trace}(\rho_{in}O^V)$. Therefore, the quantum fluctuations seen by the first observer (which have area-scaling properties), are the same as the statistical fluctuations seen by the second observer. For the latter observer, fluctuations of, say, the energy, are a measure of the heat capacity, which according to the above result is proportional to the surface area of V and not to its volume. The fluctuation-dissipation theorem

then generalizes this result for fluctuations of other operators, implying that thermodynamic quantities have area-scaling behavior [7].

Going one step further, if the volume V is chosen to be half of Minkowski space, then ρ_{in} is the density matrix for an accelerated observer [15, 16]. One can calculate the heat capacity of Rindler space radiation by the above method, yielding, again, an area dependent quantity. This is consistent with area-scaling behavior of Unruh radiation [7].

These area dependent fluctuations also give evidence for a boundary type theory. If one considers correlations between two operators $\langle O_i^{V_1} O_j^{V_2} \rangle$, then these correlations vanish when no common boundary exists—Implying that the information content of the system exists on the boundary. This line of thought is further developed in [9].

It is interesting to note that since, in general, $\langle 0|O^{V_1}O^{V_2}|0\rangle \neq \langle 0|O^{V_1}|0\rangle\langle 0|O^{V_2}|0\rangle$, then when V_2 is the complement of V_1 , we get that the vacuum is an entangled state (see [17] for a discussion of this). This implies, due to Bell inequalities, that such correlation functions cannot be reproduced by a local classical setup.

Finally, a proposed application of this area-scaling behavior was recently given by [18] to explain the Harrison-Zeldovich spectrum.

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- [1] J. D. Bekenstein, Phys. Rev. **D7**, 2333 (1973).
 - [2] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
 - [3] R. Bousso, Rev. Mod. Phys. **74**, 825 (2002), hep-th/0203101.
 - [4] M. Srednicki, Phys. Rev. Lett. **71**, 666 (1993), hep-th/9303048.
 - [5] L. Bombelli, R. K. Koul, J.-H. Lee, and R. D. Sorkin, Phys. Rev. **D34**, 373 (1986).

- [6] H. Casini, (2003), hep-th/0312238.
- [7] R. Brustein and A. Yarom, (2003), hep-th/0311029.
- [8] R. Brustein and A. Yarom, (2003), hep-th/0302186.
- [9] R. Brustein, D. H. Oaknin, and A. Yarom, (2003), hep-th/0310091.
- [10] R. Brustein, D. Eichler, S. Foffa, and D. H. Oaknin, Phys. Rev. **D65**, 105013 (2002), hep-th/0009063.
- [11] D. Oaknin, private communications.
- [12] S. Foffa, private communications.
- [13] S. Weinberg, *The Quantum Theory of Fields* volume 1 (Cambridge University Press, Cambridge, U.K., 2001).
- [14] R. P. Feynman, *Statistical Mechanics: A Set of Lectures* (Perseus Publishing, 1998).
- [15] C. Holzhey, F. Larsen, and F. Wilczek, Nucl. Phys. **B424**, 443 (1994), hep-th/9403108.
- [16] D. Kabat and M. J. Strassler, Phys. Lett. **B329**, 46 (1994), hep-th/9401125.
- [17] B. Reznik, Found. Phys. **33**, 167 (2003), quant-ph/0212044.
- [18] D. H. Oaknin, (2003), hep-ph/0308078.
- [19] We state this in order to avoid the case where the volumes are composed of infinitely many disconnected subsets. If the volumes are composed of a finite number of disconnected subsets, then this case can easily be reduced to the connected volumes case.